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A METHOD OF COMPUTING SUBSONIC FLOWS AROUND GIVEN AIRFOILS

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SUMMARY

An extremely simple computational method is established for obtaining circulatory flows around given bodies to a high degree of accuracy for flows satisfying the linear pressure and specific-volume equation of state. The method depends not on an integral equation but on the transformation from the hodograph to the physical plane involving the determination of an arbitrary analytic function. The determination of the arbitrary analytic function by elementary means results in a close approximation of the given body.

INTRODUCTION

The real mathematical problem arising from the theory of compressible fluids is the examination by the methods of function theory of the solutions of the nonlinear first-order simultaneous partial differential equations:

$$\left.\begin{aligned} \phi_x &= \frac{1}{\rho}\psi_y \\ \phi_y &= -\frac{1}{\rho}\psi_x \end{aligned}\right\} \tag{1}$$

or of the nonlinear second-order partial differential equations

$$\left(\rho\phi_x\right)_x + \left(\rho\phi_y\right)_y = 0$$

and

$$\left(\frac{1}{\rho}\psi_x\right)_x + \left(\frac{1}{\rho}\psi_y\right)_y = 0$$

where ϕ and ψ are the potential and stream functions, respectively, and x and y are the independent coordinates in the physical plane of the position of a particle of the flow.

It has been well known for half a century that a change of the independent variables from the physical coordinates x and y to the hodograph coordinates θ and q transforms the nonlinear system to a linear one, that is, equations (1) become

$$\left.\begin{aligned} \phi_\theta &= \tau_1(q)\psi_q \\ \phi_q &= -\tau_2(q)\psi_\theta \end{aligned}\right\} \tag{2}$$

where

$$\tau_1(q) = \frac{q}{\rho(q)}$$

$$\tau_2(q) = \frac{1 - M^2(q)}{q\rho(q)}$$

ρ is the density, and M the Mach number. Various methods of function theory have been established for the solutions of the linear hodograph equations (references 1 to 4). In particular the theory of Σ-monogenic functions, which is essentially a generalization of the theory of analytic functions (references 3 to 5), has been used with varied success. Analytic functions and Σ-monogenic functions appear to have strong qualitative similarities so that certain solutions of equations (2) "correspond" to well-known analytic functions.

Because of the difficulties encountered in attempting to solve boundary-value problems for Σ-monogenic functions, "approximate" qualitative solutions may be obtained more simply by a method of correspondence. The analogous boundary-value problem (the flow past a given body) is obtained for incompressible fluids in the hodograph plane, and the "corresponding" complex potential in terms of Σ-monogenic functions is then used as a solution to a qualitatively similar problem (reference 6).

This procedure yields a solution in the hodograph plane, which in general is not in desirable form. The transformation from the hodograph to the physical plane therefore becomes of basic importance. Use is made of the transformation of Gelbart (reference 7) which was also established by Lin and Germain (references 8 and 9). For the isentropic equation of state the transformation has the differential form:

$$dz = \frac{e^{i\theta}}{q}\left(d\phi + \frac{1}{\rho}\, d\psi\right) \tag{3}$$

Under the assumption that the pressure and specific-volume relation is a linear relation,[1] the hodograph equations of the flow, as in the incompressible case, are essentially reduced to the Cauchy-Riemann equations, so that ϕ and ψ in equation (3) are essentially harmonic functions. This enables the transitional equation to be written in a relatively simple form:

$$z = f(\zeta) - \frac{1}{4}\overline{\int \frac{[G'(\zeta)]^2}{f'(\zeta)}\, d\zeta} \tag{4}$$

where ζ is the position vector in an auxiliary plane, $G(\zeta)$ is the complex potential of the incompressible fluid flow around a circle in this plane, with or without circulation, $f(\zeta)$ is an arbitrary analytic function of ζ, and the bar over the integral term indicates the complex conjugate. If ζ takes on the values at the points of a streamline of a flow around the circle in the ζ-plane, equation (4) is the parametric representation of a streamline in the physical plane. The circle as a streamline in the ζ-plane is transformed into the obstacle as a streamline in the z-plane.

Since for each analytic $f(\zeta)$ there corresponds a compressible flow and vice versa, it is therefore necessary only to determine $f(\zeta)$ such that the circle in the ζ-plane maps into a given body in the z-plane by means of equation (4) in order to solve the direct problem, that is, to obtain the compressible fluid flow around a given body. This paper does not attempt to give an exact solution to this problem. The object of the paper is to find an analytic $f(\zeta)$ by elementary and simple means such that equation (4) is a mapping of the circle onto the given body to a high degree of accuracy.

This is done by considering the following maximum-minimum problem: For $|\zeta| = R$, to find an analytic $f(\zeta)$ such that the maximum value of

$$\left| f(\zeta) - \frac{1}{4}\overline{\int \frac{[G'(\zeta)]^2}{f'(\zeta)}\, d\zeta} - \omega(\zeta) \right|$$

[1]Such an assumption was first considered by Chaplygin (reference 10) and later reconsidered by Kármán, Tsien, Bers, Gelbart, and others (references 7 and 11 to 13).

shall be a minimum, where $\omega(\zeta)$ is the analytic function that maps the exterior of the circle onto the exterior of the given body, the point at infinity remaining fixed. The technique of this paper was less specifically set forth in a paper by Bartnoff and Gelbart (reference 14) and consists in expanding each side of the equation

$$f(\zeta) - \frac{1}{4}\overline{\int \frac{[G'(\zeta)]^2}{f'(\zeta)}\,d\zeta} = \omega(\zeta)$$

in a power series in $e^{i\delta}$ and solving for the coefficients of $f(\zeta)$ by equating the coefficients of the terms with like powers of $e^{i\delta}$. The simplicity of this procedure is obvious.

By setting up an integral equation and solving it by successive approximations Bers (reference 15) is able to obtain exact solutions. Though the computations are not too long they are many times longer than those presented in this paper.

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SYMBOLS

ϕ	potential function
ψ	stream function
x,y	Cartesian coordinates in physical plane
z	complex variable in physical plane
ζ	complex variable in auxiliary plane
w	complex variable in distorted hodograph plane
q	magnitude of velocity
θ	direction of velocity in physical plane
$\vec{q}$	velocity vector
$\tilde{q}$	magnitude of distorted velocity

q_∞	magnitude of velocity at infinity
$q_{i,\infty}$	velocity of undisturbed incompressible flow
p	pressure
ρ	density
a	speed of sound
γ	ratio of specific heats
M	Mach number
M_∞	stream Mach number
$G(\zeta)$	complex potential of flow around circle in ζ-plane
Ω	complex potential of compressible fluid flow
Γ	circulation of incompressible flow
α_i	angle of attack of incompressible flow around circle
α	angle of attack of compressible flow around airfoil
$f(\zeta),w(\zeta)$	analytic functions
$\omega(\zeta)$	conformal mapping of exterior of circle onto exterior of Joukowski profile
R	radius of circle in ζ-plane
δ	variable between 0 and 2π
ϵ	constant in Joukowski mapping
$A_n,\beta_0,b_n,$ B_n,D_n	auxiliary constants
k,K,P,Q	constants
τ_1,τ_2	functions of q defined by equations (2)

0,1 as subscripts, particular values of variables

$| \; |$ with symbol, absolute value

$\overline{}$ over term, complex conjugate

with symbol, total derivative

THE FUNDAMENTAL RELATIONS

The four fundamental laws governing the steady, two-dimensional, irrotational flow of an ideal fluid are: The equation of state,

$$p = k\rho^{\gamma} \tag{5}$$

where p is the pressure, ρ the density, γ the ratio of the specific heats, and k a constant; Bernoulli's equation,

$$\frac{q^2}{2} + \int \frac{dp}{\rho} = \text{Constant} \tag{6}$$

the continuity equation,

$$\operatorname{div}\left(\frac{\rho}{\rho_0}\vec{q}\right) = 0 \tag{7}$$

and the circulation equation,

$$\operatorname{curl}\left(\vec{q}\right) = 0 \tag{8}$$

Of these four laws the latter two, the continuity equation and the circulation equation, give rise to the potential function ϕ, given by

$$d\phi = q \cos\theta \, dx + q \sin\theta \, dy \tag{9}$$

where q is the magnitude of the velocity of a particle at the point x,y of the flow and θ its direction, and to the stream function ψ, given by

$$d\psi = -\rho q \sin\theta \, dx + \rho q \cos\theta \, dy \tag{10}$$

From the fact that $d\phi$ and $d\psi$ are exact differentials it follows that

$$\left.\begin{aligned}\phi_x &= \frac{1}{\rho}\,\psi_y \\ \phi_y &= -\frac{1}{\rho}\,\psi_x\end{aligned}\right\} \tag{11}$$

Multiplying each side of equation (10) by $\frac{i}{\rho}$ and adding to equation (9),

$$dz = \frac{e^{i\theta}}{q}\left(d\phi + \frac{i}{\rho}\,d\psi\right) \tag{12}$$

Equation (12) is the mapping function from the hodograph plane to the physical plane.

By differentiating equation (12) first with respect to θ and the result with respect to q and equating that with the differentiation taken in the reverse order, the hodograph equations of the flow are obtained from the real and the imaginary parts:

$$\left.\begin{aligned}\phi_\theta &= \frac{q}{\rho}\,\psi_q \\ \phi_q &= -q\,\frac{d}{dq}\left(\frac{1}{\rho q}\right)\psi_\theta\end{aligned}\right\} \tag{13}$$

From equations (5) and (6) the following relations are obtained:

$$a^2 = a_0{}^2 - \frac{\gamma - 1}{2}\,q^2 \tag{14}$$

where a is the velocity of sound with $a^2 = \frac{dp}{d\rho}$,

$$p = p_0\left(1 - \frac{\gamma - 1}{2}\,\frac{q^2}{a_0{}^2}\right)^{\frac{\gamma}{\gamma-1}} = p_0\left(1 + \frac{\gamma - 1}{2}\,\frac{q^2}{a^2}\right)^{-\frac{\gamma}{\gamma-1}} \tag{15}$$

$$\rho = \rho_0\left(1 - \frac{\gamma - 1}{2}\frac{q^2}{a_0^2}\right)^{\frac{1}{\gamma-1}} = \rho_0\left(1 + \frac{\gamma - 1}{2}\frac{q^2}{a^2}\right)^{-\frac{1}{\gamma-1}} \tag{16}$$

$$M^2 = \frac{q^2}{a^2} = -\frac{q}{\rho}\frac{d\rho}{dq} = \frac{q^2}{a_0^2 - \frac{\gamma - 1}{2} q^2} \tag{17}$$

where the subscript zero indicates the stagnation state. It is convenient to normalize the constants so that $\rho_0 = a_0 = 1$. This is equivalent to introducing the dimensionless variables ρ/ρ_0 and q/a_0.

From equation (17), equations (13) become

$$\left.\begin{aligned} \phi_\theta &= \frac{q}{\rho}\psi_q \\ \phi_q &= -\frac{1 - M^2}{\rho q}\psi_\theta \end{aligned}\right\} \tag{18}$$

Upon replacing equation (5) by

$$p - p_1 = k\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) \tag{19}$$

equations (14) to (17) become

$$a^2 = \frac{dp}{d\rho} = -\frac{k}{\rho^2} \tag{20}$$

$$p - p_1 = k\left(\sqrt{1 + q^2} - \frac{1}{\rho_1}\right) \tag{21}$$

$$\rho^2 = \frac{1}{1 + q^2} \tag{22}$$

$$M^2 = \frac{q^2}{1 + q^2} \tag{23}$$

From equations (22) and (23) it follows that

$$\frac{1 - M^2}{\rho^2} = 1 \tag{24}$$

Thus equations (18) become

$$\left.\begin{aligned} \phi_\theta &= \frac{q}{\rho} \psi_q \\ \phi_q &= -\frac{\rho}{q} \psi_\theta \end{aligned}\right\} \tag{25}$$

There is an elementary transformation that will reduce equations (25) to the Cauchy-Riemann equations. By a change of the independent variables from θ,q to $\theta,\tilde{q}$, where

$$\tilde{q} = \int_{q_\infty}^{q} \frac{dq}{q\sqrt{1 + q^2}} \tag{26}$$

and q_∞ is the magnitude of the velocity at infinity, equations (25) become

$$\left.\begin{aligned} \phi_\theta &= \psi_{\tilde{q}} \\ \phi_{\tilde{q}} &= -\psi_\theta \end{aligned}\right\} \tag{27}$$

After the integration has been performed, equation (26) becomes

$$\tilde{q} = \log \frac{Kq}{1 + \sqrt{1 + q^2}} \tag{28}$$

where

$$K = \frac{1 + \sqrt{1 + q_\infty{}^2}}{q_\infty} \tag{29}$$

From equation (12),

$$dz = \frac{e^{i\theta}}{q}\left[d\left(\frac{\Omega + \overline{\Omega}}{2}\right) + \frac{i}{\rho}\, d\left(\frac{\Omega - \overline{\Omega}}{2i}\right)\right]$$

$$= \frac{e^{i\theta}}{2}\left(\frac{1}{q} + \frac{1}{\rho q}\right) d\Omega + \frac{e^{i\theta}}{2}\left(\frac{1}{q} - \frac{1}{\rho q}\right) d\overline{\Omega} \tag{30}$$

where $\Omega = \phi + i\psi$. From equations (28) and (22),

$$dz = \frac{K}{2}e^{iw}\, d\Omega - \frac{1}{2K}e^{i\overline{w}}\, d\overline{\Omega} \tag{31}$$

where $w = \theta + i\tilde{q}$ and Ω is an analytic function of w. Thus,

$$z = \frac{K}{2}\int e^{iw}\, d\Omega - \frac{1}{2K}\overline{\int e^{-iw}\, d\Omega} \tag{32}$$

The function Ω is the complex potential of a compressible flow in the physical plane or in the hodograph plane, depending on in which plane it is being considered. However, if Ω is considered as a function of w, it is analytic and represents the complex potential of an incompressible fluid flow in the w-plane.

Equation (32) will therefore remain invariant under <u>any</u> conformal transformation $w = w(\zeta)$. By setting

$$w(\zeta) = -i \log \frac{2f'(\zeta)}{KG'(\zeta)} \tag{33}$$

equation (32) takes on the convenient form

$$z = f(\zeta) - \frac{1}{4}\overline{\int \frac{\left[G'(\zeta)\right]^2}{f'(\zeta)}\, d\zeta} \tag{34}$$

where $G(\zeta) = \Omega[w(\zeta)]$ and $f(\zeta)$ is an arbitrary analytic function of ζ. Since $w = \theta + i\tilde{q}$, the magnitude of the velocity of the compressible fluid flow can be obtained from equations (33) and (28), for

$$e^{-i\theta}e^{\tilde{q}} = \frac{K}{2}\frac{G'(\zeta)}{f'(\zeta)}, \qquad \tilde{q}(q_\infty) = 0 \tag{35}$$

and

$$q = \frac{|G'/f'|}{1 - |G'/2f'|^2} \tag{36}$$

It is clear from equation (36) that $|G'/2f'| < 1$.

It appears to be most convenient to fix $G(\zeta)$ to be the complex potential of a uniform flow past a circle of radius R in the ζ-plane with a constant circulation of magnitude Γ; that is,

$$G(\zeta) = q_{i,\infty}\left(\zeta + \frac{R^2}{\zeta}\right) - \frac{i\Gamma}{2\pi}\log\frac{\zeta}{R} \tag{37}$$

and

$$\Gamma = 4\pi q_{i,\infty} R \sin\alpha_i \tag{38}$$

where α_i is the angle of attack and $q_{i,\infty}$ is the velocity at infinity of the incompressible fluid flow past the circle in the ζ-plane. If mapping (34) is to keep the point at infinity fixed and is to be such that q_∞ is to be bounded, then $f'(\zeta)$ must have the form

$$f'(\zeta) = \sum_{n=0}^{\infty}\frac{b_n}{\zeta^n}, \qquad b_0 \neq 0 \tag{39}$$

This paper deals only with flows of this type.

Thus, from equation (35), b_0 determines the velocity of the flow at infinity. It is easily verified that the proper choice of b_1 insures that the mapping of the circle from the ζ-plane to the streamline in the z-plane will also be a closed curve. The other coefficients of $f'(\zeta)$, and thus $f(\zeta)$, may be so determined that the flow be past a preassigned obstacle.

It is this procedure that is followed in this paper.

THE DETERMINATION OF THE ARBITRARY ANALYTIC FUNCTION $f(\zeta)$

If the function $f(\zeta)$ has been fixed in equation (34), then each point in the ζ-plane exterior to the circle R is mapped into a point in the z-plane such that the velocity of the compressible flow at the point z is given by formula (36) and the boundary of the circle $|\zeta| = R$ goes into a closed streamline in the z-plane (assuming b_1 has been suitably chosen). Therefore, to solve the direct problem, that is, to obtain a uniform flow past a preassigned body, one need only determine $f(\zeta)$ such that the transformation in equation (34) transforms the circle $|\zeta| = R$ into a preassigned shape. (It is assumed in this paper that the conformal mapping of the exterior of a circle onto the exterior of any simple closed curve is given. See references 16 and 17.)

Because the left-hand side of equation (34) is not an analytic function, the mapping of the circle onto the preassigned curve is not identical, point by point, with the conformal mapping of the circle onto the preassigned curve. Denote the conformal mapping by $\omega = \omega(\zeta)$. The assumption of this paper, in order to obtain a simple and highly accurate; though not exact, means of obtaining $f(\zeta)$, is that the inverse images of the two mappings, the conformal and the one given by equation (34), of the same point on the given curve subtend a small angle at the center of the circle $\zeta = Re^{i\delta}$, $0 \leqq \delta < 2\pi$. This is undoubtedly true for airfoil shapes, as has been shown by the computations of this paper. That it also is true for curves of circular shapes has been verified in the paper by Bartnoff and Gelbart (reference 14).

On the basis of this assumption the maximum value of

$$\left| f(\zeta) - \frac{1}{4}\overline{\int \frac{[G'(\zeta)]^2}{f'(\zeta)}\, d\zeta} - \omega(\zeta) \right| \tag{40}$$

is small for $\zeta = Re^{i\delta}$. By equating the coefficients of like powers of $e^{i\delta}$ in $z(\zeta)$ and in $\omega(\zeta)$, and thus determining $f(\zeta)$, it is assumed that the terms neglected in this process will be small. This has been verified for the cases chosen in this paper and in reference 14.

The process here mentioned will now be applied to the case of a symmetric Joukowski airfoil for a given angle of attack α and a flow whose speed at infinity is q_∞. The fact that a symmetric Joukowski airfoil is chosen instead of a general airfoil is by no means a necessary restriction. The same method with the same simplicity can be applied to all airfoil shapes.

To satisfy the Kutta-Joukowski condition at the trailing edge, the flow past the circle $\zeta = Re^{i\delta}$ with an angle of attack α_i or a circulation

$$\Gamma = -4\pi q_{i,\infty} R \sin \alpha_i \tag{41}$$

is given by

$$G(\zeta) = q_{i,\infty}\left(\zeta e^{-i\alpha_i} + \frac{R^2}{\zeta e^{-i\alpha_i}}\right) - \frac{i\Gamma}{2\pi} \log \frac{\zeta e^{-i\alpha_i}}{R} \tag{42}$$

The stagnation points on the circle occur at $\delta = 0^\circ$ and $\delta = 180^\circ + 2\alpha_i$. Taking the derivative of $G(\zeta)$ and substituting equation (41) for Γ,

$$G'(\zeta) = q_{i,\infty} e^{-i\alpha_i}\left(1 + \frac{2iR \sin \alpha_i}{\zeta e^{-i\alpha_i}} - \frac{R^2}{\zeta^2 e^{-2i\alpha_i}}\right) \tag{43}$$

Squaring each side of equation (43),

$$\left[G'(\zeta)\right]^2 = q_{i,\infty}{}^2 e^{-2i\alpha_i}\left(A_0 + A_1 \frac{1}{\zeta} + A_2 \frac{1}{\zeta^2} + A_3 \frac{1}{\zeta^3} + A_4 \frac{1}{\zeta^4}\right)$$

where

$$\left.\begin{aligned}
A_0 &= 1 \\
A_1 &= 4iRe^{i\alpha_i} \sin \alpha_i \\
A_2 &= -2R^2 e^{2i\alpha_i}\left(1 + 2 \sin^2\alpha_i\right) \\
A_3 &= -4iR^3 e^{3i\alpha_i} \sin \alpha_i \\
A_4 &= R^4 e^{4i\alpha_i}
\end{aligned}\right\} \tag{44}$$

Recall that $f'(\zeta) = \sum_{n=0}^{\infty} \frac{b_n}{\zeta^n}$. Let

$$\frac{1}{f'(\zeta)} = \sum_{n=0}^{\infty} \frac{B_n}{\zeta^n} \tag{45}$$

Hence

$$1 = \left(\sum_{n=0}^{\infty} \frac{B_n}{\zeta^n}\right)\left(\sum_{n=0}^{\infty} \frac{b_n}{\zeta^n}\right) \tag{46}$$

so that

$$\left.\begin{aligned} 1 &= b_0 B_0 \\ 0 &= b_n B_0 + \cdot \cdot \cdot + b_0 B_n, \qquad n > 0 \end{aligned}\right\} \tag{47}$$

Equations (47) can be solved for B_n:

$$\left.\begin{aligned} B_0 &= \frac{1}{b_0} \\ B_n &= -\frac{1}{b_0}\left(b_n B_0 + \cdot \cdot \cdot + b_1 B_{n-1}\right) \end{aligned}\right\} \tag{48}$$

For later use B_1 and B_2 in terms of the unknown constants b_n are given:

$$\left.\begin{aligned} B_1 &= -\frac{b_1}{b_0^{\,2}} \\ B_2 &= -\frac{b_2}{b_0^{\,2}} + \frac{b_1^{\,2}}{b_0^{\,3}} \end{aligned}\right\} \tag{49}$$

Now it is possible to write

$$\frac{\left[G'(\zeta)\right]^2}{f'(\zeta)} = q_{i,\infty}^{\;\;2} e^{-2i\alpha_i} \sum_{n=0}^{\infty} D_n \frac{1}{\zeta^n} \tag{50}$$

where

$$D_n = A_0B_n + \cdot \cdot \cdot + A_4B_{n-4} \tag{51}$$

Thus,

$$-\overline{\int\frac{\left[G'(\zeta)\right]^2}{f'(\zeta)}\,d\zeta} = -q_{i,\infty}{}^2 e^{2i\alpha_i}\left(\overline{D}_0\overline{\zeta} + \overline{D}_1 \log \overline{\zeta} - \sum_{n=1}^{\infty}\frac{\overline{D}_{n+1}}{n\overline{\zeta}^n}\right) \tag{52}$$

and

$$f(\zeta) = b_{-1} + b_0\zeta + b_1 \log \zeta - \sum_{n=1}^{\infty}\frac{b_{n+1}}{n\zeta^n} \tag{53}$$

On the circle $\zeta = Re^{i\delta}$, equation (34) becomes

$$z = b_{-1} + b_0Re^{i\delta} + b_1 \log R + b_1(i\delta) - \sum_{n=1}^{\infty}\frac{b_{n+1}}{nR^n}e^{-in\delta} -$$

$$\frac{q_{i,\infty}{}^2 e^{2i\alpha_i}}{4}\left(\overline{D}_0Re^{-i\delta} + \overline{D}_1 \log R - \overline{D}_1(i\delta) - \sum_{n=1}^{\infty}\frac{\overline{D}_{n+1}}{nR^n}e^{in\delta}\right) \tag{54}$$

The Joukowski transformation that maps the exterior of a circle of radius R onto the exterior of an airfoil is given by

$$\omega(\zeta) = -\epsilon + \zeta + \frac{(R - \epsilon)^2}{\zeta - \epsilon}$$

$$= -\epsilon + \zeta + \sum_{n=1}^{\infty}\frac{(R - \epsilon)^2\epsilon^{n-1}}{\zeta^n} \tag{55}$$

where ϵ is the center of the circle that is mapped onto a strip. On the circle $\zeta = Re^{i\delta}$, equation (55) becomes

$$\omega\left(Re^{i\delta}\right) = -\epsilon + Re^{i\delta} + \sum_{n=1}^{\infty}\frac{(R - \epsilon)^2\epsilon^{n-1}}{R^n}e^{-in\delta} \tag{56}$$

Equating coefficients of like terms in equations (54) and (56), the following simultaneous equations for b_n are obtained:

$$-\epsilon = b_{-1} + b_1 \log R - \frac{q_{i,\infty}^2 e^{2i\alpha_i}}{4} \bar{D}_1 \log R \tag{57}$$

$$R = Rb_0 + \frac{q_{i,\infty}^2 e^{2i\alpha_i}}{4R} \bar{D}_2 \tag{58}$$

$$0 = b_1 + \frac{q_{i,\infty}^2 e^{2i\alpha_i}}{4} \bar{D}_1 \tag{59}$$

$$\frac{(R - \epsilon)^2}{R} = -\frac{b_2}{R} - \frac{q_{i,\infty}^2 e^{2i\alpha_i}}{4} R\bar{D}_0 \tag{60}$$

and

$$\frac{(R - \epsilon)^2 \epsilon^{n-1}}{R^n} = -\frac{b_{n+1}}{nR^n}, \qquad n \geqq 2 \tag{61}$$

Since D_n is a function of b_0, b_1, . . ., b_n only, equations (58), (59), and (60) can be solved for b_0, b_1, and b_2. Once these are obtained, equation (57) can be trivially solved for b_{-1}. Equation (61) can also be trivially solved for b_n, $n \geqq 3$. It remains then only to solve equations (58), (59), and (60) for b_0, b_1, and b_2.

Equation (59), it should be pointed out, is the condition on b_1 that the mapping of the circle $\zeta = Re^{i\delta}$ by equation (34) be a closed curve.

Equations (58), (59), and (60) are definitely not linear equations in b_0 or $\bar{b}_0$. But if, from equation (35), the value of $q_{i,\infty}$ in terms of b_0 is substituted into the three equations they become linear in β_0, b_1, and b_2, where β_0 is the absolute value of b_0, that is, $\beta_0 = |b_0|$.

For the point at infinity equation (35) gives rise to the relation

$$e^{-i\alpha} = \frac{K}{2} \frac{q_{i,\infty} e^{-i\alpha_i}}{b_0} \tag{62}$$

or

$$q_{i,\infty} = \frac{2}{K} b_0 e^{-i\left(\alpha-\alpha_i\right)} = 2Pb_0 e^{-i\left(\alpha-\alpha_i\right)} \tag{63}$$

where α is the angle of attack of the compressible fluid flow past the airfoil, and

$$P = \frac{1}{K} = \frac{q_\infty}{1 + \sqrt{1 + q_\infty^{\ 2}}} \tag{64}$$

$$0 < P < 1 \tag{65}$$

Since $q_{i,\infty}$ and P are real, the argument of b_0 must be $\alpha - \alpha_i$, or $b_0 = \beta_0 e^{i\left(\alpha-\alpha_i\right)}$. Equation (63) then becomes

$$q_{i,\infty} = 2P\beta_0 \tag{66}$$

After substituting equation (66) and $\overline{D}_0 = \dfrac{1}{\overline{b}_0}$ into equation (60),

$$b_2 = -(R - \epsilon)^2 - P^2\beta_0 R^2 e^{i\left(\alpha+\alpha_i\right)} \tag{67}$$

Now

$$\begin{aligned} \overline{D}_1 &= \overline{A}_0\overline{B}_1 + \overline{A}_1\overline{B}_0 \\ &= -\frac{\overline{b}_1}{\overline{b}_0^{\ 2}} - \frac{4iRe^{-i\alpha_i} \sin \alpha_i}{\overline{b}_0} \end{aligned} \tag{68}$$

The substitution of this and equation (66) into equation (59) yields

$$b_1 - P^2 e^{2i\alpha} \overline{b}_1 - 4iP^2 Re^{i\alpha} \beta_0 \sin \alpha_i = 0 \tag{69}$$

The complex conjugate of equation (69) results in

$$\overline{b}_1 = P^2 e^{-2i\alpha} b_1 - 4iP^2 Re^{-i\alpha} \beta_0 \sin \alpha_i \tag{70}$$

and by substituting equation (70) into equation (69), equation (69) becomes

$$b_1 - P^4 b_1 + 4iP^4 Re^{i\alpha} \beta_0 \sin \alpha_i - 4iP^2 Re^{i\alpha} \beta_0 \sin \alpha_i = 0 \tag{71}$$

Hence

$$b_1 = \frac{4iP^2 Re^{i\alpha} \sin \alpha_i}{1 + P^2} \beta_0 \tag{72}$$

Proceeding as before,

$$\overline{D}_2 = \overline{A}_0 \overline{B}_2 + \overline{A}_1 \overline{B}_1 + \overline{A}_2 \overline{B}_0$$

$$= -\frac{\overline{b}_2}{\overline{b}_0^{\,2}} + \frac{\overline{b}_1^{\,2}}{\overline{b}_0^{\,3}} + \frac{4iRe^{-i\alpha_i} \overline{b}_1 \sin \alpha_i}{\overline{b}_0^{\,2}} -$$

$$\frac{2R^2 e^{-2i\alpha_i}\left(1 + 2 \sin^2 \alpha_i\right)}{\overline{b}_0} \tag{73}$$

The substitution of this as well as equations (72) and (67) into equation (58) yields a linear equation in β_0:

$$R^2 = \beta_0 e^{i(\alpha-\alpha_i)} R^2 + P^2 e^{2i\alpha}(R - \epsilon)^2 + P^4 \beta_0 R^2 e^{i(\alpha-\alpha_i)} - \frac{16P^6 R^2 e^{i(\alpha-\alpha_i)} \beta_0 \sin^2\alpha_i}{\left(1 + P^2\right)^2} + \frac{16P^4 R^2 e^{i(\alpha-\alpha_i)} \beta_0 \sin^2\alpha_i}{1 + P^2} - 2P^2 R^2 \beta_0 e^{i(\alpha-\alpha_i)}\left(1 + 2\sin^2\alpha_i\right) \tag{74}$$

Thus the solution for β_0 can be written

$$\beta_0 = \frac{R^2 - P^2 e^{2i\alpha}(R - \epsilon)^2}{e^{i(\alpha-\alpha_i)} R^2 \left(1 - P^2\right)^2 \left[1 - \left(\frac{2P \sin\alpha_i}{1 + P^2}\right)^2\right]} \tag{75}$$

In determining the coefficients of $f(\zeta)$ it is desirable to do so in terms of q_∞ and α, which are to be considered as preassigned. The quantity α_i must then be considered as an unknown. The four equations for β_0, b_1, b_2, and α_i are then equations (58), (59), (60), and (63). Since β_0 must be real, α_i can be obtained by setting the imaginary part of equation (75) equal to zero:

$$\text{Im}\left\{\left[R^2 - P^2 e^{2i\alpha}(R - \epsilon)^2\right] e^{-i(\alpha-\alpha_i)}\right\} = 0 \tag{76}$$

or

$$-R^2 \sin\left(\alpha - \alpha_i\right) - P^2(R - \epsilon)^2 \sin\left(\alpha + \alpha_i\right) = 0 \qquad (77)$$

Hence

$$\frac{\sin\left(\alpha - \alpha_i\right)}{\sin\left(\alpha + \alpha_i\right)} = -\frac{P^2(R - \epsilon)^2}{R^2} = -Q \qquad (78)$$

It is seen from equation (78) that Q is positive. From equation (78)

$$\frac{\sin \alpha_i \cos \alpha - \cos \alpha_i \sin \alpha}{\sin \alpha_i \cos \alpha + \cos \alpha_i \sin \alpha} = Q \qquad (79)$$

and

$$1 - \frac{\tan \alpha}{\tan \alpha_i} = Q\left(1 + \frac{\tan \alpha}{\tan \alpha_i}\right) \qquad (80)$$

Then for α_i:

$$\tan \alpha_i = \frac{1 + Q}{1 - Q} \tan \alpha \qquad (81)$$

where

$$Q = \left[\frac{P(R - \epsilon)}{R}\right]^2 < 1 \qquad (82)$$

Since $0 \leqq \alpha < \frac{\pi}{2}$, α_i always has a solution between 0 and $\frac{\pi}{2}$ and $\alpha_i \geqq \alpha$. When $\alpha = 0$, $\alpha_i = 0$.

To obtain b_n, $n \geqq 3$, equation (61) is solved:

$$b_n = -(n - 1)(R - \epsilon)^2 \epsilon^{n-2}, \qquad n \geqq 3 \qquad (83)$$

Finally,

$$f'(\zeta) = b_0 + b_1\frac{1}{\zeta} + \frac{b_2}{\zeta^2} + \sum_{n=3}^{\infty} \frac{b_n}{\zeta^n}$$

$$= b_0 + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} - \sum_{n=3}^{\infty} \frac{(R - \epsilon)^2(n - 1)\epsilon^{n-2}}{\zeta^n}$$

$$= b_0 + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + (R - \epsilon)^2\left[\frac{1}{\zeta^2} - \frac{1}{(\zeta - \epsilon)^2}\right] \tag{84}$$

Also,

$$f(\zeta) = b_{-1} + b_0\zeta + b_1 \log \zeta - \frac{b_2 + (R - \epsilon)^2}{\zeta} + \frac{(R - \epsilon)^2}{\zeta - \epsilon} \tag{85}$$

where b_{-1}, b_0, b_1, and b_2 are given by the equations (57), (75), (72), and (67).

The fact that a Joukowski airfoil was chosen is in no way a limitation of the method. The same procedure could be followed with the same simplicity for the conformal mapping function $\omega(\zeta) = A_{-1}\zeta + \sum_{n=0}^{\infty} A_n/\zeta^n$ of the exterior of a circle onto the exterior of an arbitrary airfoil shape, the point at infinity remaining fixed and the value of the derivative at infinity being bounded.

If nonsymmetric Joukowski airfoils are considered, there is no essential change in the representation of b_n. Here, ϵ would be complex, but at no point in the work has it been essential that ϵ be real.

It should be remarked that the method of this paper does not yield an exact solution of the direct problem, but for practical purposes gives a very good approximation to the exact solution. However, the results obtained by this method are exact for the flow around a body that can be computed readily and for which there is a precise mathematical expression, such as the Joukowski airfoil used in the example. Bers (reference 15) approaches this problem by means of an integral equation, which yields exact results whenever the iteration process is valid, but the computations are much more involved.

RESULTS OF COMPUTATIONS

Computations have been carried out for a given Joukowski profile having the following geometric characteristics: $\epsilon = 0.15$ and $R = 1.15$. The constants of the transformation for free-stream Mach numbers of 0.5 and 0.685 are given in table 1. The velocity distributions calculated for the actual profiles for various angles of attack and the same two values of the Mach number are given in tables 2 and 3 and plotted in figures 1 and 2. Tables 4 and 5 present the pressure distributions corresponding to the velocity distributions of tables 2 and 3.

The velocity computed at the trailing edge ($\zeta = R$) is zero unavoidably because $G'(\zeta)$ is zero at this point in the formula

$$q = \frac{\left|\frac{G'(\zeta)}{f'(\zeta)}\right|}{1 - \left|\frac{G'(\zeta)}{2f'(\zeta)}\right|^2} \tag{86}$$

and $f'(\zeta)$ is not zero. The exact $f(\zeta)$ which maps a circle onto a Joukowski profile would have a zero derivative, but the one obtained by the present method is only an approximation. Accordingly, the data for $\delta = 0^\circ$ and 360° have been omitted from tables 2 to 5 and figures 1 and 2.

ACCURACY

The methods of this paper are primarily for computational purposes and are to be judged by the simplicity and accuracy of the computations.

The body for which the exact flow is given by the work of this paper is represented by the equation

$$z = \omega(\zeta) + \frac{1}{4} q_{i,\infty}{}^2 \sum_{n=2}^{\infty} \frac{\overline{D}_{n+1}}{n\overline{\zeta}^n} \tag{87}$$

where $\omega(\zeta)$ is the mapping function of the given body. The accuracy then depends on how small the second term of the right-hand side of equation (87) is on the circle $\zeta = Re^{i\delta}$. The ordinates for the given body and for the one obtained from equation (87) are given in table 6 for $M_\infty = 0.685$ and $\alpha_i = 0^\circ$; the corresponding profiles are given in figure 3. For this case, the series term in equation (87) is an alternating series of decreasing terms which begins with the term involving $\bar{D}_3$. The error in neglecting the remainder is then always less than its first term. The maximum value of the term involving $\bar{D}_{12}$ did not exceed 0.0002. The sum $\sum_{n=2}^{11} \frac{\bar{D}_{n+1}}{n\bar{\zeta}^n}$ on $\zeta = Re^{i\delta}$ was computed and nowhere exceeded 0.022.

The mapping function $f(\zeta)$ to be determined for a given flow around a given body not only depends on the given body but also on the free-stream velocity and the angle of attack. This implies that the body obtained from mapping (87) varies slightly for different free-stream flow, although $\omega(\zeta)$ remains fixed.

In table 3 and figure 2 of the present paper and also in the paper by Bers (reference 15) are data for the velocity distribution of the flow around a Joukowski airfoil, for which $\epsilon = 0.15$, $R = 1.15$, $M_\infty = 0.685$, and $\alpha_i = 2^\circ\ 27'$. Although a precise comparison cannot be made since there is no precise way of fixing the points of comparison, the two results appear to compare quite favorably.

Syracuse University
Syracuse, N. Y., October 27, 1947

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TABLE 1.— CONSTANTS OF THE TRANSFORMATION

$M_\infty = 0.5$; $q_\infty = 0.5774$; $P = 0.2679$; $\epsilon = 0.15$; $R = 1.15$					
α_i	0°	5°	10°	15°	20°
α	0°	$4^\circ\ 29'$	$8^\circ\ 59'$	$13^\circ\ 31'$	$18^\circ\ 5'$
β_0	1.0976	1.0934	1.1063	1.1243	1.1431
$\lvert b_1 \rvert / R$	0.0000	0.0255	0.0515	0.0780	0.1048
$\lvert 1 + b_2 \rvert / R^2$	0.0788	0.0785	0.0794	0.0807	0.0821
$q_{i,\infty}$	0.5881	0.5858	0.5928	0.6024	0.6125

$M_\infty = 0.685$; $q_\infty = 0.9402$; $P = 0.3963$; $\epsilon = 0.15$; $R = 1.15$					
α_i	0°	$2^\circ\ 27'$	5°	10°	15°
α	0°	$2^\circ\ 43'$	$3^\circ\ 56'$	$7^\circ\ 54'$	$11^\circ\ 55'$
β_0	1.2403	1.2390	1.2351	1.2619	1.2966
$\lvert b_1 \rvert / R$	0.0000	0.0405	0.0585	0.1190	0.1823
$\lvert 1 + b_2 \rvert / R^2$	0.1949	0.1946	0.1940	0.1982	0.2037
$q_{i,\infty}$	0.9831	0.9820	0.9789	1.0002	1.0277

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TABLE 2.– VELOCITY DISTRIBUTION q/q_∞ FOR VARIOUS ANGLES OF ATTACK

$[M_\infty = 0.5; \ \epsilon = 0.15; \ R = 1.15]$

δ \ α_1	0°	5°	10°	15°	20°
10	0.875	0.871	0.867	0.855	0.840
20	.889	.895	.901	.902	.896
30	.912	.926	.941	.950	.954
40	.943	.967	.991	1.009	1.021
50	.982	1.016	1.050	1.079	1.101
60	1.027	1.073	1.119	1.161	1.195
70	1.077	1.137	1.199	1.255	1.328
80	1.129	1.207	1.287	1.362	1.430
90	1.182	1.283	1.385	1.484	1.575
100	1.234	1.362	1.492	1.621	1.743
110	1.281	1.443	1.611	1.777	1.941
120	1.319	1.526	1.741	1.960	2.179
130	1.342	1.608	1.887	2.177	2.478
140	1.339	1.684	2.054	2.449	2.874
150	1.285	1.744	2.249	2.809	3.444
160	1.126	1.749	2.458	3.297	4.332
170	.734	1.550	2.542	3.839	5.685
180	.000	.875	1.955	3.526	6.218
190	.734	.000	.792	1.813	3.423
200	1.126	.560	.000	.609	1.362
210	1.285	.852	.431	.000	.467
220	1.339	1.002	.674	.343	.000
230	1.342	1.077	.816	.553	.283
240	1.319	1.107	.989	.685	.466
250	1.281	1.110	.942	.768	.587
260	1.234	1.096	.960	.817	.668
270	1.182	1.071	.961	.844	.720
280	1.129	1.040	.950	.855	.753
290	1.077	1.005	.934	.856	.771
300	1.027	.970	.914	.851	.781
310	.982	.938	.894	.844	.787
320	.943	.910	.876	.837	.791
330	.912	.888	.864	.834	.797
340	.889	.873	.858	.837	.809
350	.875	.869	.864	.850	.832

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TABLE 3.— VELOCITY DISTRIBUTION q/q_∞ FOR VARIOUS ANGLES OF ATTACK

$[M_\infty = 0.685; \quad \epsilon = 0.15; \quad R = 1.15]$

δ \ α_1	0°	$2^\circ\ 27'$	5°	10°	15°
10	0.876	0.870	0.866	0.858	0.843
20	.894	.897	.896	.904	.907
30	.919	.929	.931	.949	.963
40	.953	.970	.976	1.004	1.028
50	.996	1.022	1.029	1.069	1.106
60	1.045	1.079	1.092	1.146	1.197
70	1.100	1.145	1.162	1.234	1.303
80	1.159	1.218	1.241	1.334	1.425
90	1.219	1.295	1.326	1.445	1.566
100	1.279	1.375	1.416	1.569	1.727
110	1.333	1.456	1.510	1.707	1.913
120	1.377	1.535	1.605	1.860	2.133
130	1.403	1.608	1.700	2.034	2.401
140	1.400	1.666	1.789	2.235	2.744
150	1.338	1.691	1.860	2.475	3.212
160	1.157	1.627	1.864	2.746	3.902
170	.731	1.320	1.637	2.908	4.875
180	.000	.583	.893	2.280	5.092
190	.731	.224	.000	.856	2.453
200	1.157	.741	.565	.000	.684
210	1.338	1.013	.874	.442	.000
220	1.400	1.146	1.036	.696	.358
230	1.403	1.205	1.116	.846	.576
240	1.377	1.219	1.148	.931	.714
250	1.333	1.207	1.149	.977	.800
260	1.279	1.179	1.132	.994	.851
270	1.219	1.140	1.102	.992	.877
280	1.159	1.096	1.066	.979	.887
290	1.100	1.051	1.026	.959	.885
300	1.045	1.007	.987	.936	.878
310	.996	.967	.952	.913	.868
320	.953	.932	.920	.893	.860
330	.919	.905	.896	.879	.856
340	.894	.885	.881	.873	.859
350	.876	.876	.877	.878	.873

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TABLE 4.– PRESSURE DISTRIBUTION p/p_0 FOR VARIOUS ANGLES OF ATTACK

$[M_\infty = 0.5; \quad \epsilon = 0.15; \quad R = 1.15]$

δ \ α_1	0°	5°	10°	15°	20°
10	1.120	1.119	1.118	1.115	1.111
20	1.124	1.126	1.127	1.127	1.126
30	1.130	1.134	1.138	1.141	1.142
40	1.139	1.145	1.152	1.157	1.161
50	1.150	1.159	1.169	1.178	1.185
60	1.163	1.176	1.191	1.204	1.215
70	1.178	1.196	1.216	1.235	1.260
80	1.194	1.219	1.246	1.272	1.297
90	1.211	1.244	1.281	1.317	1.352
100	1.228	1.272	1.320	1.370	1.419
110	1.244	1.302	1.366	1.433	1.502
120	1.257	1.333	1.418	1.510	1.607
130	1.265	1.365	1.479	1.606	1.746
140	1.264	1.395	1.551	1.732	1.938
150	1.245	1.419	1.639	1.905	2.226
160	1.193	1.421	1.736	2.151	2.694
170	1.086	1.342	1.776	2.432	3.432
180	1.000	1.120	1.508	2.268	3.727
190	1.086	1.000	1.100	1.448	2.215
200	1.193	1.051	1.000	1.060	1.272
210	1.245	1.115	1.030	1.000	1.036
220	1.264	1.155	1.073	1.019	1.000
230	1.265	1.178	1.106	1.050	1.013
240	1.257	1.187	1.126	1.075	1.036
250	1.244	1.188	1.138	1.094	1.056
260	1.228	1.184	1.143	1.106	1.072
270	1.211	1.176	1.144	1.112	1.083
280	1.194	1.166	1.141	1.115	1.090
290	1.178	1.156	1.136	1.116	1.095
300	1.163	1.146	1.131	1.114	1.097
310	1.150	1.137	1.125	1.112	1.098
320	1.139	1.130	1.121	1.111	1.099
330	1.130	1.124	1.118	1.110	1.101
340	1.124	1.120	1.116	1.111	1.104
350	1.120	1.119	1.118	1.114	1.109

TABLE 5.– PRESSURE DISTRIBUTION p/p_0 FOR VARIOUS ANGLES OF ATTACK

$[M_\infty = 0.685; \quad \epsilon = 0.15; \quad R = 1.15]$

δ \ α_1	0°	2° 27'	5°	10°	15°
10	1.295	1.292	1.290	1.285	1.276
20	1.306	1.308	1.308	1.312	1.314
30	1.322	1.328	1.329	1.340	1.349
40	1.343	1.354	1.357	1.375	1.391
50	1.370	1.387	1.391	1.418	1.443
60	1.402	1.424	1.433	1.470	1.505
70	1.439	1.469	1.481	1.532	1.582
80	1.479	1.520	1.537	1.604	1.672
90	1.521	1.576	1.598	1.687	1.780
100	1.564	1.635	1.665	1.782	1.907
110	1.603	1.695	1.736	1.891	2.058
120	1.636	1.756	1.810	2.015	2.241
130	1.656	1.812	1.886	2.158	2.469
140	1.653	1.858	1.957	2.327	2.766
150	1.607	1.878	2.015	2.532	3.182
160	1.477	1.827	2.018	2.769	3.802
170	1.213	1.594	1.836	2.911	4.692
180	1.000	1.140	1.306	2.365	4.891
190	1.213	1.022	1.000	1.283	2.514
200	1.477	1.219	1.133	1.000	1.189
210	1.607	1.381	1.294	1.083	1.000
220	1.653	1.470	1.396	1.195	1.055
230	1.656	1.511	1.449	1.278	1.137
240	1.636	1.521	1.471	1.329	1.204
250	1.603	1.513	1.472	1.358	1.251
260	1.564	1.493	1.460	1.369	1.281
270	1.521	1.466	1.440	1.368	1.296
280	1.479	1.436	1.416	1.359	1.302
290	1.439	1.406	1.390	1.346	1.301
300	1.402	1.377	1.365	1.332	1.297
310	1.370	1.351	1.342	1.318	1.291
320	1.343	1.330	1.322	1.306	1.286
330	1.322	1.313	1.308	1.297	1.284
340	1.306	1.301	1.298	1.294	1.286
350	1.295	1.296	1.296	1.296	1.294

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TABLE 6.— THE JOUKOWSKI AND ACTUAL PROFILES[1]

$\left[M_\infty = 0.685; \quad \alpha_i = 0^\circ\right]$

δ	x_J	x_A	y_J	y_A
0	2.000	2.022	0.000	0.000
10	1.960	1.980	.001	.009
20	1.842	1.859	.008	.023
30	1.654	1.664	.025	.045
40	1.407	1.409	.055	.077
50	1.114	1.108	.097	.117
60	.788	.775	.147	.164
70	.442	.423	.200	.211
80	.089	.068	.251	.255
90	−.262	−.282	.295	.291
100	−.599	−0.616	.326	.314
110	−.915	−0.926	.342	.325
120	−1.203	−1.205	.340	.320
130	−1.457	−1.452	.319	.300
140	−1.672	−1.660	.280	.266
150	−1.843	−1.827	.225	.219
160	−1.968	−1.955	.158	.160
170	−2.044	−2.038	.081	.086
180	−2.069	−2.068	.000	.000

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[1]Joukowski: $z_J = x_J + iy_J$

Actual: $z_A = x_A + iy_A$

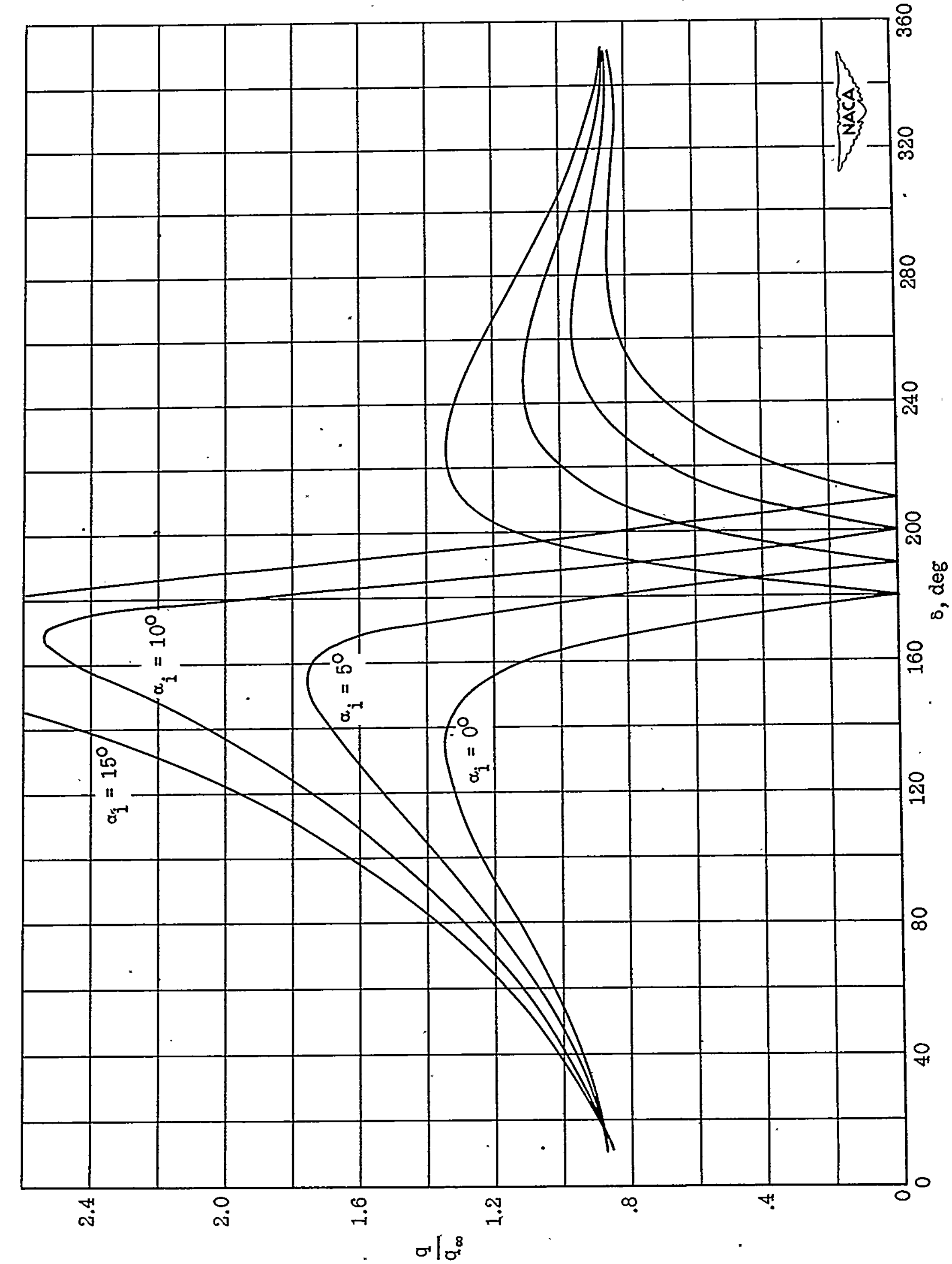

Figure 1.- Velocity distribution. $M_{\infty} = 0.5$; $\epsilon = 0.15$; $R = 1.15$.

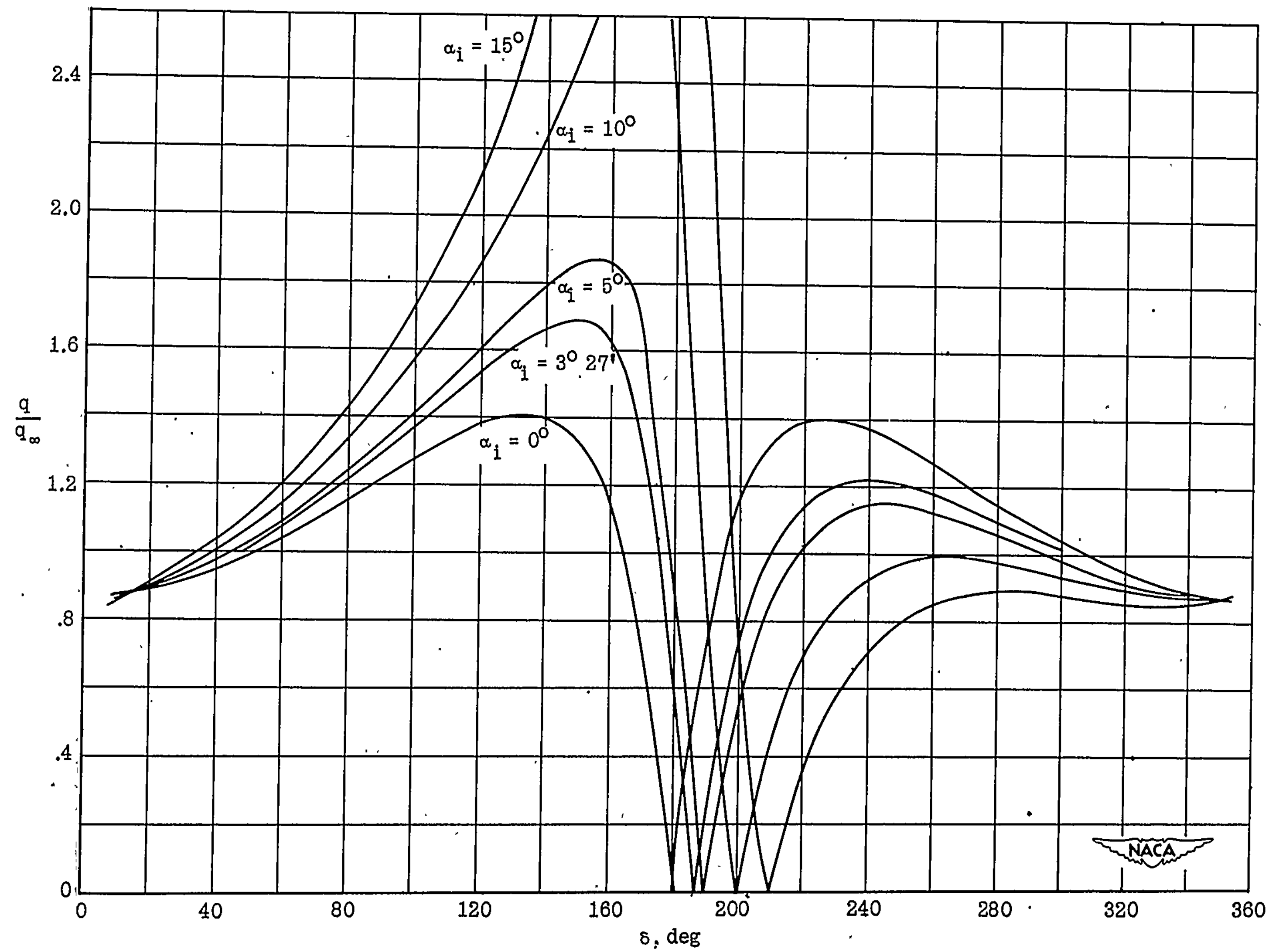

Figure 2.- Velocity distribution. $M_\infty = 0.685$; $\epsilon = 0.15$; $R = 1.15$.

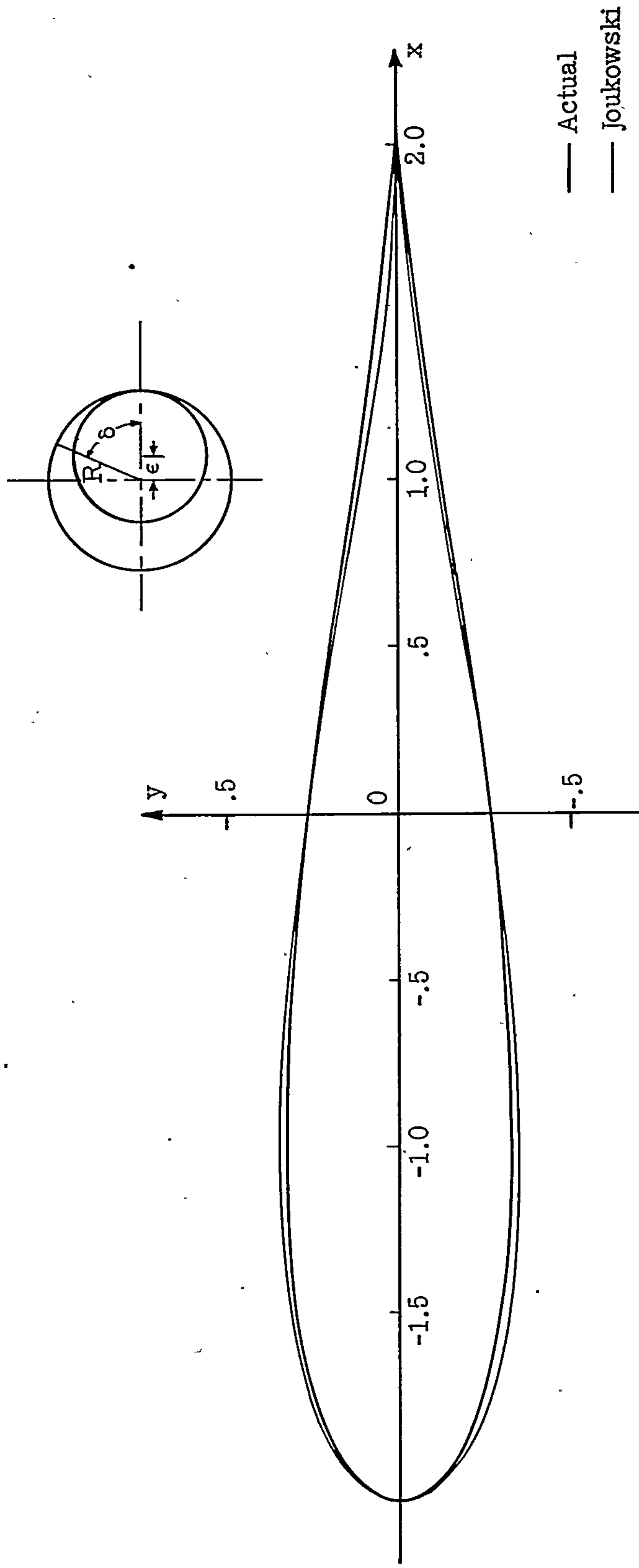

Figure 3.- The actual and Joukowski profiles.